

# From pseudoholomorphic functions to the associated real manifold

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## Abstract

This paper studies first the differential inequalities that make it possible to build a global theory of pseudoholomorphic functions in the case of one or several complex variables. In the case of one complex dimension, we prove that the differential inequalities describing pseudoholomorphicity can be used to define a one-real-dimensional manifold (by the vanishing of a function with nonzero gradient), which is here a 1-parameter family of plane curves. On studying the associated envelopes, such a parameter can be eliminated by solving two nonlinear partial differential equations. The classical differential geometry of curves can be therefore exploited to get a novel perspective on the equations describing the global theory of pseudoholomorphic functions.

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## I. INTRODUCTION

The progress in complex analysis and differential geometry has led to many important concepts in pure mathematics and mathematical physics from the nineteenth century until recent times. For example, when twistor theory and its applications to general relativity were developed by Penrose [1], and his school, the subject of complex general relativity [2] emerged as a fascinating branch of modern mathematical physics, where the tools of complex differential geometry were applied to find self-dual or anti-self-dual solutions of (vacuum) Einstein equations [3], and also to develop a suitable definition of twistor in curved spacetime [1], without relying on the differential equation that defines Killing spinor fields, but rather considering suitable surfaces (e.g. the totally null  $\alpha$ - and  $\beta$ -surfaces [4]).

The very concept of complex manifold involves complex-analytic transition functions. More precisely, a complex manifold is meant to be a paracompact Hausdorff space covered by neighbourhoods each homeomorphic to an open set in  $C^m$  ( $m = 1, 2, \dots$ ), such that, where these neighbourhoods overlap, the local coordinates transform by a holomorphic transformation. Thus, if  $z^1, \dots, z^m$  are complex local coordinates in one such neighbourhood, and if  $w^1, \dots, w^m$  are local coordinates in another neighbourhood, where they are both defined one has  $w^i = w^i(z^1, \dots, z^m)$  and each  $w^i$  is a holomorphic function (see below) of the  $z$ 's, and the determinant  $\partial(w^1, \dots, w^m)/\partial(z^1, \dots, z^m)$  does not vanish. Well known examples of this abstract concept are the space  $C^m$ , complex projective space  $CP^m$ , non-singular submanifolds of  $CP^m$ , the complex torus, the orientable surfaces [5].

From the point of view of complex analysis, the assumption of holomorphic transition functions is nontrivial. By definition, the function

$$f : z = x + iy \rightarrow f(z) = u(x, y) + iv(x, y)$$

is holomorphic if it is a continuous function of the complex variable  $z$ , for which the first derivative  $f'(z)$  exists. This is enough to ensure continuity of  $f'(z)$  as well [6, 7], jointly with the many properties that one learns in introductory courses, including the equivalence with the Weierstrass definition of complex-analytic function of a complex variable, that involves absolute and uniform convergence of a power series in the first place. However, if the assumption of differentiability is no longer made, one can define a one-parameter family of functions of a complex variable, that are holomorphic only if the parameter  $\mu$  in the definition is set to 1. These are the pseudo-holomorphic functions, beautifully presented by

Bers [8], but we here rely upon the Caccioppoli approach [9, 10], better suited if one wants to build a global theory, but apparently (much) less known in the literature, maybe because of lack of an English translation. Interestingly, in the theory of pseudoanalytic functions, no use is made of analytic functions' theory, but on the contrary important topics of the latter, e.g. the Picard theorem, appear in a new light, through the analysis of qualitative aspects. In other words, one can obtain elementary proofs of classical theorems, revealing their intimate nature of metric and topological properties. The Cauchy-Riemann differential equations

$$u_x = v_y, \quad u_y = -v_x$$

that express the holomorphic nature of a function are then replaced, in the pseudoholomorphic case, by differential inequalities, i.e. simple majorizations replace the equalities among angles of the conformal representation.

Section II defines pseudoholomorphic functions of a complex variable according to Caccioppoli, studying in detail some basic equations in such a definition. Section III presents our definition of pseudoholomorphic function of several (i.e. two or more) complex variables, inspired by the Caccioppoli work in the case of a single complex variable. Since such functions are believed to be more fundamental, one may hope that the manifolds one arrives at by means of them are also more fundamental in a suitable sense. After a review of discwise quasi-conformal maps with  $n$  complex variables in Sect. 4, we study in Sect. 5 the 1-parameter family of plane curves associated with a pseudoholomorphic function, while the envelopes for such curves are considered in Sect. 6. Concluding remarks and open problems are presented in Sect. 7, while some technical points are discussed in the appendix.

## II. PSEUDOHOLOMORPHIC FUNCTIONS OF A COMPLEX VARIABLE

Let  $z$  be the familiar notation for complex variable  $z = x + iy$ ,  $x \in \mathbf{R}, y \in \mathbf{R}$ , and let  $w$  be the continuous function with image

$$w(z) = u(x, y) + iv(x, y) \tag{2.1}$$

defined in a field  $A$ , a bounded open set of the  $z$  plane such that all its points are internal points. Let the functions  $u(x, y), v(x, y)$  satisfy the following assumptions:

(i)  $u(x, y)$  and  $v(x, y)$  are absolutely continuous in  $x$  and  $y$  for almost all values of  $y$  and  $x$  respectively, while their first derivatives  $u_x, u_y, v_x, v_y$  are square-integrable in every internal portion of  $A$ .

(ii) If

$$J = \frac{\partial(u, v)}{\partial(x, y)} = u_x v_y - u_y v_x \quad (2.2)$$

is the Jacobian of the map  $(x, y) \rightarrow (u, v)$ ,  $\Phi(x, y)$  is the upper limit

$$\Phi(x, y) \equiv \overline{\lim}_{\Delta z \rightarrow 0} \left| \frac{\Delta w}{\Delta z} \right|, \quad (2.3)$$

$\varphi(x, y)$  is the lower limit

$$\varphi(x, y) \equiv \underline{\lim}_{\Delta z \rightarrow 0} \left| \frac{\Delta w}{\Delta z} \right|, \quad (2.4)$$

then  $J \geq 0$  almost everywhere in  $A$ , and there exists a positive real number  $\mu \in ]0, 1]$  such that

$$\varphi(x, y) \geq \mu \Phi(x, y) \quad (2.5)$$

almost everywhere in  $A$ . The function  $w$  is here said to be pseudoholomorphic<sup>1</sup>. of parameter  $\mu$  [9, 10] Every value of  $\mu \leq 1$  corresponds to a class  $C_\mu$  of pseudoholomorphic functions; in particular, if  $\mu = 1$ ,  $C_1$  is the class of holomorphic functions.

Outside of the holomorphic framework, the increment ratio  $\frac{\Delta w}{\Delta z}$  has indeed a rich structure because, upon defining

$$m \equiv \frac{\Delta y}{\Delta x} \quad (2.6)$$

one has

$$\frac{\Delta w}{\Delta z} = \frac{u_x + iv_x + m(u_y + iv_y)}{(1 + im)} + \frac{(\varepsilon_1 + i\varepsilon_2) + m(\varepsilon_3 + i\varepsilon_4)}{(1 + im)}, \quad (2.7)$$

where  $\varepsilon_i$  tends to zero as  $\Delta z \rightarrow 0$ , for all  $i = 1, 2, 3, 4$ . Multiplication and division by  $(1 - im)$  on the right-hand side yields therefore

$$\frac{\Delta w}{\Delta z} = A + iB + O(\varepsilon), \quad (2.8)$$

where

$$A \equiv \frac{(u_x + mu_y) + m(v_x + mv_y)}{(1 + m^2)}, \quad (2.9)$$

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<sup>1</sup> The work in Refs. [9, 10] uses actually the nomenclature *pseudoanalytic*, but we prefer to speak of pseudoholomorphic functions, to avoid confusion with the local theory of pseudoanalytic functions, which relies instead upon generalized Cauchy-Riemann equations [11]

$$B \equiv \frac{(v_x + mv_y) - m(u_x + mu_y)}{(1 + m^2)}, \quad (2.10)$$

and hence

$$\lim_{\Delta z \rightarrow 0} \left| \frac{\Delta w}{\Delta z} \right|^2 = A^2 + B^2. \quad (2.11)$$

Now a patient calculation shows that, in the expression  $(1 + m^2)^2(A^2 + B^2)$ , a cancellation occurs and some reassembling can be made, so that eventually [12]

$$A^2 + B^2 = (1 + m^2)^{-1}(E + 2Fm + Gm^2) \equiv r(m), \quad (2.12)$$

where, according to a standard notation, we have set

$$E \equiv (u_x)^2 + (v_x)^2, \quad G \equiv (u_y)^2 + (v_y)^2, \quad F \equiv u_x u_y + v_x v_y. \quad (2.13)$$

To study the maximum or minimum of  $A^2 + B^2$  as a function of the real variable  $m$ , we have to evaluate its first derivative, which vanishes if  $m$  solves the algebraic equation of second degree

$$Fm^2 + (E - G)m - F = 0, \quad (2.14)$$

solved by

$$m = -\frac{(E - G)}{2F} \pm \frac{1}{2F} \sqrt{(E - G)^2 + 4F^2} = m_{\pm}. \quad (2.15)$$

It is clear from (2.15) that for any generic E, F and G,  $m_+$  is positive whereas  $m_-$  is negative. Once we take second derivative of the function  $A^2 + B^2$  with respect to  $m$ , after a little algebra, we obtain the simple expression

$$\frac{\partial^2}{\partial m^2}(A^2 + B^2) = -\frac{2F}{m(1 + m^2)} \quad (2.16)$$

and hence it is clear that  $\frac{\partial^2}{\partial m^2}(A^2 + B^2)|_{m_+} < 0$  whereas  $\frac{\partial^2}{\partial m^2}(A^2 + B^2)|_{m_-} > 0$ .

Interestingly, setting for convenience

$$\omega \equiv \sqrt{(E - G)^2 + 4F^2} = \sqrt{(E + G)^2 - 4J^2}, \quad (2.17)$$

one finds from (2.12) that [12]

$$r_+ = r(m_+) = \frac{E + 2Fm_+ + Gm_+^2}{(1 + m_+^2)} = \frac{(E + G)}{2} + \frac{\omega}{2}, \quad (2.18)$$

$$r_- = r(m_-) = \frac{E + 2Fm_- + Gm_-^2}{(1 + m_-^2)} = \frac{(E + G)}{2} - \frac{\omega}{2}. \quad (2.19)$$

In other words, the maximum and minimum values of  $r$  are themselves solutions of the quadratic equation [12]

$$\rho^2 - (E + G)\rho + J^2 = 0, \quad (2.20)$$

and the upper and lower limit (2.3) and (2.4) turn out to obey the identities [9, 10]

$$2\Phi^2 = E + G + \omega, \quad 2\varphi^2 = E + G - \omega, \quad (2.21)$$

$$\Phi^2 + \varphi^2 = E + G, \quad \Phi^2\varphi^2 = \frac{1}{4}[(E + G)^2 - \omega^2] = J^2 = EG - F^2, \quad (2.22)$$

jointly with the inequalities

$$\mu J \leq \varphi^2 \leq \Phi^2 \leq \frac{J}{\mu}, \quad J \geq \frac{\mu}{(1 + \mu^2)}(\Phi^2 + \varphi^2). \quad (2.23)$$

By virtue of assumption (i), one can build a sequence  $(u_n(x, y), v_n(x, y))$  of pairs of functions, describing flat surfaces, such that [9]

$$\lim_{n \rightarrow \infty} u_n(x, y) = u(x, y), \quad \lim_{n \rightarrow \infty} v_n(x, y) = v(x, y), \quad (2.24)$$

uniformly in every closed portion of the field  $A$ , and such that their partial derivatives with respect to  $x$  and  $y$  converge in mean of order 2;  $u_n$  and  $v_n$  being functions as smooth as desired, or even polynomials. Hence it follows that on the surface  $S$  associated to  $u(x, y)$  and  $v(x, y)$ , areas and lengths have the classical expressions. This means that, if  $D$  and  $L$  are a domain and a line within  $A$ , respectively, the area  $\tau D$  and the length  $\tau L$  read as

$$\int \int_D J \, dx \, dy = \int \int_D \sqrt{EG - F^2} \, dx \, dy, \quad (2.25)$$

$$\tau L = \int_L \sqrt{E dx^2 + 2F dx \, dy + G dy^2}. \quad (2.26)$$

With this nomenclature,  $\tau$  is the plane transformation of  $z$  into  $w$  defined by the equations

$$\tau : u = u(x, y), \quad v = v(x, y)$$

which in turn describe a flat surface  $S$  carried by the plane  $w$ . Such a map  $\tau$  is said to be a pseudo-conformal transformation with parameter  $\mu$ , i.e. a pseudo-conformal representation of  $S$  upon  $A$ . To every point  $z_0$  of the field  $A$  there corresponds a point  $P(z_0)$  of  $S$ , having as trace on the plane  $w$  the point  $w_0 = w(z_0)$ .

### A. Quasi-conformal maps for real and complex manifolds

For the case of real manifolds we recall, following Ref. [13], that for any pseudo-group of homeomorphisms of Euclidean space one can define the corresponding category of manifolds. Thus, the full pseudo-group of homeomorphisms, the subgroup of smooth diffeomorphisms, the pseudo-group of quasiconformal maps and the pseudo-group of Lipschitz maps give rise to the theory of topological manifolds,  $C^\infty$  manifolds, quasi-conformal manifolds and Lipschitz manifolds, respectively. In particular, a homeomorphism  $\varphi : D \rightarrow \mathbf{R}^n$  is  $K$ -quasiconformal if, for all  $x \in D$ ,

$$\limsup_{r \rightarrow 0} \frac{\max |\varphi(y) - \varphi(x)|}{\min |\varphi(y) - \varphi(x)|} \leq K,$$

with

$$|y - x| = \sqrt{\sum_{k=1}^n (y_k - x_k)^2} = r.$$

The map  $\varphi$  is quasiconformal if it is  $K$ -quasiconformal for some  $K \geq 1$ . This range of values of  $K$  is responsible for distortion of relative distances of nearby points by a bounded factor.

On the other hand, for functions of complex variable, the concept of quasiconformal maps [14] was introduced by Grötzsch, who considered homeomorphisms (2.1) with a positive Jacobian (2.2). Such a map takes infinitesimal circles (cf. the above remarks on distortion of relative distances) into infinitesimal ellipses, and is called *quasi-conformal* if the eccentricity of these ellipses is uniformly bounded. This condition can be expressed analytically by either of the three equivalent differential inequalities in (2.23) (our  $\mu$  parameter is the inverse of the  $Q$  parameter used by Bers). This property is conformally invariant: if  $w = w(z)$  has it, so does the function  $U(\zeta) = F\{w[f(\zeta)]\}$ , where  $F$  and  $f$  are conformal mappings.

Bers [14] calls a function  $w(z)$  as in (2.1) quasi-conformal if it is of the form

$$w(z) = f[\chi(z)], \tag{2.27}$$

where  $\chi$  is a quasi-conformal homeomorphism and  $f$  is an holomorphic function. This definition is suggested by an important result of Morrey [15] according to which, if  $w(z)$  is a quasi-conformal function defined in the unit disk, then it admits the representation (2.27) where  $\zeta = \chi(z)$  is a homeomorphism of the set  $|z| \leq 1$  onto the set  $|\zeta| \leq 1$  with  $\chi(0) = 0, \chi(1) = 1$ , which satisfies together with its inverse  $\chi^{-1}$  a uniform Holder condition, and where  $f(\zeta)$  is a holomorphic function of the complex variable  $\zeta$ ,  $|\zeta| < 1$ .

It is a profound result of the work in Ref. [14], which was inspired also by the work of Mori [16], that the geometric definition relying upon Eq. (2.27) is equivalent to the analytic definition given at the beginning of this section, inspired by Morrey, Caccioppoli, Bers and Nirenberg, according to which a continuous function  $w(z) = u(x, y) + iv(x, y)$  in a domain  $D$  is quasi-conformal if it has  $L_2$  derivatives satisfying the differential inequalities (2.23) almost everywhere. By equivalent we here mean that the geometric and analytic definition imply each other [14]. Note that, in light of the results in Refs. [9, 10, 14], our pseudoholomorphic functions are also quasi-conformal maps.

### III. PSEUDOHOLOMORPHIC FUNCTIONS OF $n$ COMPLEX VARIABLES

Let us proceed now by generalizing the results of previous section to  $n$  complex variables.

For  $n$  complex variables  $z_1, z_2, \dots, z_n$  with  $z_k = x_k + iy_k, \forall k = 1, 2, \dots, n$   $x_k \in \mathbf{R}, y_k \in \mathbf{R}$ , let  $w$  be the continuous function with image

$$w(z_1, z_2, \dots, z_n) = u(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) + iv(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n). \quad (3.1)$$

Let the functions  $u(\{x_k\}, \{y_k\}), v(\{x_k\}, \{y_k\})$  satisfy the following assumptions:

(i)  $u(\{x_k\}, \{y_k\})$  and  $v(\{x_k\}, \{y_k\})$  are absolutely continuous in  $\{x_k\}$  and  $\{y_k\}$  for almost all values of  $\{y_k\}, \{x_k\}$  respectively, while their first derivatives  $\{u_{x_k}\}, \{u_{y_k}\}, \{v_{x_k}\}, \{v_{y_k}\}$  are square-integrable in every internal portion of the domain of definition.

(ii) If

$$J_k = \frac{\partial(u, v)}{\partial(x_k, y_k)} = u_{x_k} v_{y_k} - u_{y_k} v_{x_k}, \quad (3.2)$$

$\Phi_k$  is the upper limit

$$\Phi_k \equiv \overline{\lim}_{\Delta z_k \rightarrow 0} \left| \frac{\Delta w}{\Delta z_k} \right|, \quad (3.3)$$

$\varphi_k$  is the lower limit

$$\varphi_k \equiv \underline{\lim}_{\Delta z_k \rightarrow 0} \left| \frac{\Delta w}{\Delta z_k} \right|, \quad (3.4)$$

then  $J_k \geq 0$  almost everywhere in  $A$ , and there exists a positive real number  $\mu_k \in ]0, 1]$  such that

$$\varphi_k \geq \mu_k \Phi_k \quad (3.5)$$

almost everywhere in  $A$ . We then say that the function  $w$  is pseudoholomorphic of parameter  $\mu_k$  for all  $k = 1, 2, \dots, n$ .



Outside of the holomorphic framework, the increment ratio  $\frac{\Delta w}{\Delta z_k}$  has indeed a rich structure as we know already from Sect. 2 because, upon defining

$$m_k \equiv \frac{\Delta y_k}{\Delta x_k} \quad (3.6)$$

one has  $n$  increment ratios

$$\frac{\Delta w}{\Delta z_k} = \frac{u_{x_k} + i v_{x_k} + m_k(u_{y_k} + i v_{y_k})}{(1 + i m_k)} + \frac{(\varepsilon_1 + i \varepsilon_2)_k + m_k(\varepsilon_3 + i \varepsilon_4)_k}{(1 + i m_k)}, \quad (3.7)$$

where  $(\varepsilon_i)_k$  tends to zero as  $\Delta z_k \rightarrow 0$ , for all  $i = 1, 2, 3, 4$  and for all  $k = 1, 2, \dots, n$ .

Multiplication and division by  $(1 - i m_k)$  on the right-hand side of (3.7) yields therefore

$$\frac{\Delta w}{\Delta z_k} = A_k + i B_k + O(\varepsilon), \quad (3.8)$$

where

$$A_k \equiv \frac{(u_{x_k} + m_k u_{y_k}) + m_k(v_{x_k} + m_k v_{y_k})}{(1 + m_k^2)}, \quad (3.9)$$

$$B_k \equiv \frac{(v_{x_k} + m_k v_{y_k}) - m_k(u_{x_k} + m_k u_{y_k})}{(1 + m_k^2)}, \quad (3.10)$$

and hence

$$\lim_{\Delta z_k \rightarrow 0} \left| \frac{\Delta w}{\Delta z_k} \right|^2 = A_k^2 + B_k^2. \quad (3.11)$$

With the help of the same calculations leading to Eq. (2.12) we now find

$$A_k^2 + B_k^2 = (1 + m_k^2)^{-1}(E_k + 2F_k m_k + G_k m_k^2) \equiv r(m_k), \quad (3.12)$$

where, according to our own notation, we have set

$$E_k \equiv (u_{x_k})^2 + (v_{x_k})^2, \quad G_k \equiv (u_{y_k})^2 + (v_{y_k})^2, \quad F_k \equiv u_{x_k} u_{y_k} + v_{x_k} v_{y_k}. \quad (3.13)$$

To study the maximum or minimum of  $A_k^2 + B_k^2$  as a function of  $m_k$ , we have to evaluate its first derivative, which vanishes if  $m_k$  solves the algebraic equation of second degree

$$F_k m_k^2 + (E_k - G_k) m_k - F_k = 0, \quad (3.14)$$

solved by

$$m_k = -\frac{(E_k - G_k)}{2F_k} \pm \frac{1}{2F_k} \sqrt{(E_k - G_k)^2 + 4F_k^2} = (m_{\pm})_k. \quad (3.15)$$

It is clear from (3.15) that for any generic  $E_k$ ,  $F_k$  and  $G_k$ ,  $(m_+)_k$  is positive whereas  $(m_-)_k$  is negative. And then once we take second derivative of the function  $A_k^2 + B_k^2$  with respect to  $m_k$ , we find, as in Eq. (2.16),

$$\frac{\partial^2}{\partial m_k^2}(A_k^2 + B_k^2) = -\frac{2F_k}{m_k(1 + m_k^2)}, \quad (3.16)$$

and hence it is clear that  $\frac{\partial^2}{\partial m_k^2}(A_k^2 + B_k^2)|_{(m_+)_k} < 0$  and  $\frac{\partial^2}{\partial m_k^2}(A_k^2 + B_k^2)|_{(m_-)_k} > 0$ .

Interestingly, setting for convenience

$$\omega_k \equiv \sqrt{(E_k - G_k)^2 + 4F_k^2} = \sqrt{(E_k + G_k)^2 - 4J_k^2}, \quad (3.17)$$

one finds from (3.12) that (cf. (2.18) and (2.19))

$$(r_+)_k = r((m_+)_k) = \frac{E_k + 2F_k(m_+)_k + G_k(m_+^2)_k}{(1 + (m_+^2)_k)} = \frac{(E_k + G_k)}{2} + \frac{\omega_k}{2}, \quad (3.18)$$

$$(r_-)_k = r((m_-)_k) = \frac{E_k + 2F_k(m_-)_k + G_k(m_-^2)_k}{(1 + (m_-^2)_k)} = \frac{(E_k + G_k)}{2} - \frac{\omega_k}{2}. \quad (3.19)$$

In other words, the maximum and minimum values of  $r_k$  are themselves solutions of the quadratic equation (cf. Eq. (2.20))

$$\rho^2 - (E_k + G_k)\rho + J_k^2 = 0, \quad (3.20)$$

and the upper and lower limit (3.3) and (3.4) turn out to obey the relations (cf. (2.21)-(2.23))

$$2\Phi_k^2 = E_k + G_k + \omega_k, \quad 2\varphi_k^2 = E_k + G_k - \omega_k, \quad (3.21)$$

$$\Phi_k^2 + \varphi_k^2 = E_k + G_k, \quad \Phi_k^2 \varphi_k^2 = \frac{1}{4}[(E_k + G_k)^2 - \omega_k^2] = J_k^2 = E_k G_k - F_k^2, \quad (3.22)$$

$$\mu_k J_k \leq \varphi_k^2 \leq \Phi_k^2 \leq \frac{J_k}{\mu_k}, \quad J_k \geq \frac{\mu_k}{(1 + \mu_k^2)}(\Phi_k^2 + \varphi_k^2). \quad (3.23)$$

As one can see, for the  $n$ -variable case the functions  $u$  and  $v$  are severely constrained, since there is an  $n$ -tuple of conditions to be satisfied.

#### IV. DISCWISE QUASI-CONFORMAL MAPS WITH $n$ COMPLEX VARIABLES

In the attempt of providing examples of the functions fulfilling our definition in Sect. 3, we here present a review of part of the work in Ref. [17], devoted to the investigation of quasi-conformal functions of several complex variables.

A function  $f$  of  $n$  complex variables  $z_1, \dots, z_n$  defined in a domain  $D$  is said to be a *discwise-quasi-conformal function* with dilatation  $K = \frac{1}{\mu}$  if

- (i)  $f$  is of class  $C^1$  in  $D$ , and
- (ii)  $f$  is  $K$ -quasi-conformal (i.e. quasi-conformal with dilatation parameter  $K = \frac{1}{\mu}$ ) on each holomorphic plane.

The meaning of condition (ii) is as follows. Once we have a linear map

$$z_1 = a_1 t + b_1, \dots, z_n = a_n t + b_n, \quad (4.1)$$

the  $a$ 's and  $b$ 's being complex constants while  $t$  is a complex variable, defined on the unit disc  $\{|t| < 1\}$  and whose image lies completely in  $D$ , the composite function

$$f(t) = f(a_1 t + b_1, \dots, a_n t + b_n) \quad (4.2)$$

is always a  $K$ -quasi-conformal function in the unit disc  $\{|t| < 1\}$ . The function  $f$  is then said to be  *$K$ -discwise-quasi-conformal*, following Ref. [17]. In particular, a 1-discwise-quasi-conformal function is a holomorphic function in  $D$ . A number of important properties are found to hold, and they are as follows [17].

**Theorem 4.1.** If  $f$  is discwise quasi-conformal in a bounded domain  $D$  and continuous also on the boundary  $\partial D$  of  $D$ , then the maximum principle holds, according to which

$$\sup \{|f|; D\} = \sup \{|f|; \partial D\}.$$

**Theorem 4.2.** If, on the unit disc  $\{|t| < 1\}$ , one considers the holomorphic map

$$z_1 = \varphi_1(t), \dots, z_n = \varphi_n(t), \quad (4.3)$$

whose image lies completely in  $D$ , the composite function

$$\hat{f}(t) = f(\varphi_1(t), \dots, \varphi_n(t)) \quad (4.4)$$

is a  $K$ -quasi-conformal function in the unit disc. As a corollary, the concept of  $K$ -discwise-quasi-conformal function is invariant under holomorphic transformations. This means that, if  $f(z_1, \dots, z_n)$  is  $K$ -discwise-quasi-conformal in  $D$ , and if

$$z_1 = \varphi_1(w_1, \dots, w_m), \dots, z_n = \varphi_n(w_1, \dots, w_m) \quad (4.5)$$

is a holomorphic transformation from a domain  $B$  in  $(w_1, \dots, w_m)$ -space into  $D$ , then the composite function

$$F(w_1, \dots, w_m) = f(\varphi_1(w_1, \dots, w_m), \dots, \varphi_n(w_1, \dots, w_m)) \quad (4.6)$$

is again  $K$ -discwise-quasi-conformal in  $B$ . One can therefore define a  $K$ -discwise-quasi-conformal function as a smooth function which is  $K$ -quasi-conformal on every holomorphic surface. This holds not only in a domain  $D$ , but also for an arbitrary set, in particular on an analytic subset in the  $(z_1, \dots, z_n)$ -space.

Another peculiar property is that the sum of two  $K$ -discwise-quasi-conformal functions is not always  $K$ -discwise-quasi-conformal. For example, each of the functions

$$2z_1 + 2z_2 + \bar{z}_1 + \bar{z}_2 \text{ and } -z_1 - z_2,$$

is  $K$ -discwise-quasi-conformal, but their sum, being equal to

$$2(\operatorname{Re} z_1 + \operatorname{Re} z_2),$$

is not  $K$ -discwise-quasi-conformal, because quasi-conformal functions cannot take only real values (having to provide an open mapping) unless they are a constant.

**Theorem 4.3.** At every ordinary point, where at least one of the first partial derivatives with respect to  $z_1, \dots, z_n$  does not vanish, the real and imaginary parts of a  $K$ -discwise-quasi-conformal function  $f = u + iv$  satisfy the following system of partial differential equations identically:

$$\frac{\partial(u, v)}{\partial(x_j, x_k)} = \frac{\partial(u, v)}{\partial(y_j, y_k)}, \quad \frac{\partial(u, v)}{\partial(x_j, y_k)} = \frac{\partial(u, v)}{\partial(x_k, y_j)}, \quad j, k = 1, 2, \dots, n, \quad (4.7)$$

where the notation for the independent variables is the same as in (3.1).

**Theorem 4.4.** If  $f$  is a  $K$ -discwise-quasi-conformal function of two complex variables  $z_1, z_2$  in a domain  $D$ , then at every ordinary point  $(z_1^0, z_2^0)$  the set given by the equation

$$f(z_1, z_2) = f(z_1^0, z_2^0) = \text{constant} \quad (4.8)$$

is a two-dimensional holomorphic surface.

**Theorem 4.5.** If  $\kappa$  is a given continuous function of  $2n$  real variables  $x_1, y_1, \dots, x_n, y_n$  in a domain  $D$ , and if the modulus of  $\kappa$  is bounded by a constant  $k_0 < 1$ , then a solution of

class  $C^1$  in  $D$  of the system of partial differential equations (4.7) yields a  $K$ -discwise-quasi-conformal function  $f = u + iv$  in  $D$ , whose dilatation  $K$  is given by

$$K = \frac{(1 + k_0)}{(1 - k_0)}, \quad |\kappa| \leq k_0 < 1. \quad (4.9)$$

The results here recalled are helpful in understanding properties and limits of the quasi-conformal pseudoholomorphic framework that we are investigating, and can be compared with the different perspectives considered in Refs. [18–22].

## V. THE REAL MANIFOLD ASSOCIATED WITH A PSEUDOHOLOMORPHIC FUNCTION

Since the conditions (2.23) are differential inequalities, it is at first sight problematic to define a pseudoholomorphic manifold, even just in the case of one complex dimension. However, we may point out that, if  $\varphi^2 > \mu J$  is fulfilled (we rule out the case of equalities, for which we refer the reader to the Appendix), then also  $\Phi^2 > \mu J$  is fulfilled, because  $\Phi^2 > \varphi^2$ . Moreover, for the same reason, if  $\Phi^2 < \frac{J}{\mu}$  holds, *a fortiori*  $\varphi^2 < \frac{J}{\mu}$  holds as well. We can therefore consider three positive-definite functions  $\alpha, \beta$  and  $\gamma$  depending on  $(\mu; x, y)$ , such that (2.23) is re-expressed in the form

$$\varphi^2 - \mu J = \alpha > 0, \quad (5.1)$$

$$\frac{J}{\mu} - \Phi^2 = \beta > 0, \quad (5.2)$$

$$J - \frac{\mu}{(1 + \mu^2)}(\Phi^2 + \varphi^2) = \gamma > 0. \quad (5.3)$$

Note also that (5.2) leads to  $\mu\Phi^2 = J - \mu\beta$ , while (5.1) implies that  $\mu\varphi^2 = \mu^2J + \mu\alpha$ . Thus, on the left-hand side of (5.3), we obtain eventually exact cancellation of terms proportional to  $J$ , so that

$$-\frac{\mu(\alpha - \beta)}{(1 + \mu^2)} = \gamma > 0 \implies \alpha(\mu; x, y) < \beta(\mu; x, y). \quad (5.4)$$

We can now exploit (2.17) and (2.21) to re-express (5.1) and (5.2) in the form

$$E + G - \sqrt{(E + G)^2 - 4J^2} - 2\mu J - 2\alpha(\mu; x, y) = 0, \quad (5.5)$$

$$2\frac{J}{\mu} - E - G - \sqrt{(E + G)^2 - 4J^2} - 2\beta(\mu; x, y) = 0. \quad (5.6)$$

Within this framework, the desired pseudoholomorphic manifold is defined *implicitly* by the nonlinear equations (5.5) and (5.6) (actually by a single equation equivalent to them, see below), a procedure which is often used in simpler cases [23] in the literature. Upon using (2.2) and (2.13), we here look for  $u(x, y)$ ,  $v(x, y)$ ,  $\alpha(\mu, x, y)$  and  $\beta(\mu, x, y)$  such that (5.5) and (5.6) hold, and the consistency condition (5.4) is fulfilled. At this stage, we can express  $\sqrt{(E + G)^2 - 4J^2}$  from Eq. (5.5), and then insert the result into Eq. (5.6). This leads to a nonlinear equation without square root, reading as

$$\frac{(\mu^2 + 1)}{\mu} J - E - G + \beta - \alpha = 0, \quad (5.7)$$

which is nothing but Eq. (5.3). In particular, we may look for  $u(x, y)$  and  $v(x, y)$  in the form

$$u(x, y) = Ax^2 + Bxy + Cy^2, \quad (5.8)$$

$$v(x, y) = \tilde{A}x^2 + \tilde{B}xy + \tilde{C}y^2. \quad (5.9)$$

This ansatz turns Eq. (5.7) into the form

$$A_\mu x^2 + B_\mu xy + C_\mu y^2 = (\alpha - \beta) < 0, \quad (5.10)$$

where the coefficients are given by

$$A_\mu \equiv 2 \frac{(\mu^2 + 1)}{\mu} (A\tilde{B} - B\tilde{A}) - (4A^2 + 4\tilde{A}^2 + B^2 + \tilde{B}^2), \quad (5.11)$$

$$B_\mu \equiv 4 \left[ \frac{(\mu^2 + 1)}{\mu} (A\tilde{C} - C\tilde{A}) - (A + C)B - (\tilde{A} + \tilde{C})\tilde{B} \right], \quad (5.12)$$

$$C_\mu \equiv 2 \frac{(\mu^2 + 1)}{\mu} (B\tilde{C} - C\tilde{B}) - (4C^2 + 4\tilde{C}^2 + B^2 + \tilde{B}^2). \quad (5.13)$$

For simplicity, we may study first the particular case when (5.10) is fulfilled with  $B_\mu = 0$ ,  $A_\mu < 0$ ,  $C_\mu < 0$ . Indeed,  $B_\mu$  vanishes if  $A = -C$  and  $\tilde{A} = -\tilde{C}$ , and in such a case  $A_\mu = C_\mu$ . The quadratic form in (5.10) is then negative-definite if  $A, \tilde{A}, B, \tilde{B}$  are chosen so as to satisfy the condition

$$A\tilde{B} - B\tilde{A} < \frac{\mu}{2(\mu^2 + 1)} (4A^2 + 4\tilde{A}^2 + B^2 + \tilde{B}^2). \quad (5.14)$$

In the general case, the condition (5.10) is fulfilled if the quadratic form  $-A_\mu x^2 - B_\mu xy - C_\mu y^2$  is positive-definite. For this purpose, we have to study the eigenvalues  $\lambda$  which are the roots

of the following equation of second degree:

$$\det \begin{pmatrix} (-A_\mu - \lambda) & -\frac{1}{2}B_\mu \\ -\frac{1}{2}B_\mu & (-C_\mu - \lambda) \end{pmatrix} = 0. \quad (5.15)$$

This equation is solved by the two roots

$$\lambda = -\frac{1}{2}(A_\mu + C_\mu) \pm \frac{1}{2}\sqrt{\delta_{ABC}} = \lambda_{1,2}, \quad (5.16)$$

where

$$\delta_{ABC} = (A_\mu + C_\mu)^2 - 4 \left( A_\mu C_\mu - \frac{1}{4}B_\mu^2 \right) = (A_\mu - C_\mu)^2 + B_\mu^2 > 0, \quad (5.17)$$

and  $\lambda_1$  (resp.  $\lambda_2$ ) corresponds to the  $+$  (resp.  $-$ ) sign in front of  $\sqrt{\delta_{ABC}}$ . Since  $\delta_{ABC}$  is a sum of squares of real numbers, we are guaranteed that both roots are real. Moreover,

$$\lambda_1 - \lambda_2 = \sqrt{\delta_{ABC}} > 0, \quad (5.18)$$

and hence positivity of  $\lambda_2$  ensures positivity of  $\lambda_1$  as well. By virtue of (5.16), positivity of  $\lambda_2$  implies that

$$-\sqrt{\delta_{ABC}} > (A_\mu + C_\mu) \implies (A_\mu + C_\mu) < 0 \implies \sqrt{\delta_{ABC}} < |A_\mu + C_\mu|. \quad (5.19)$$

In other words, (5.10) is always fulfilled provided that the following two conditions hold:

$$(A_\mu + C_\mu) < 0, \quad (5.20)$$

$$\sqrt{(A_\mu - C_\mu)^2 + B_\mu^2} < |A_\mu + C_\mu|, \quad (5.21)$$

where  $A_\mu, B_\mu, C_\mu$  are displayed in (5.11)-(5.13).

We have still to understand whether our Eq. (5.10) can be seen to define a manifold. For this purpose, we recall a basic theorem [24], according to which, for a set  $V$  of an affine space  $E$  to be an hypersurface (i.e. a manifold of dimension  $N - 1$ ) of class  $C^m$  of  $E$ , it is necessary and sufficient that,  $\forall a \in V$ , there exists a neighbourhood  $U(a)$  of  $a$  within  $E$  and a scalar function  $H$ , defined in  $U$  and of class  $C^m$ , such that, in a reference system, the partial derivatives  $\frac{\partial H}{\partial x^i}(a), i = 1, 2, \dots, N$  are not all simultaneously vanishing. In our case, the scalar function pertaining to Eq. (5.10) is the 1-parameter family

$$H(\mu; x, y) \equiv A_\mu x^2 + B_\mu xy + C_\mu y^2 + \beta(\mu; x, y) - \alpha(\mu; x, y), \quad (5.22)$$

with gradient having components

$$\frac{\partial H}{\partial x} = 2A_\mu x + B_\mu y + \frac{\partial}{\partial x}(\beta - \alpha), \quad (5.23)$$

$$\frac{\partial H}{\partial y} = B_\mu x + 2C_\mu y + \frac{\partial}{\partial y}(\beta - \alpha). \quad (5.24)$$

We may therefore obtain a one-dimensional real manifold, i.e. a curve  $\Gamma$  defined implicitly as the set

$$\Gamma \equiv \{a = (x, y) : (\text{grad } H)(a) \neq 0\}. \quad (5.25)$$

The gradient of  $(\beta - \alpha)$  plays a crucial role in ensuring that the gradient of  $H$  does not vanish (otherwise we would have the singular point  $(x = 0, y = 0)$ , and hence no manifold exists).

Since Eq. (5.10) is just a particular case of Eq. (5.7), we should also consider the general form of Eq. (5.22), i.e.

$$H(\mu; x, y) \equiv (u_x)^2 + (v_x)^2 + (u_y)^2 + (v_y)^2 - \frac{(\mu^2 + 1)}{\mu}(u_x v_y - u_y v_x) + \alpha - \beta. \quad (5.26)$$

We are therefore looking for a 1-parameter family of curves defined as in (5.25), where the gradient of  $H$  has components

$$\begin{aligned} \frac{\partial H}{\partial x} &= \left(2u_x - \frac{(\mu^2 + 1)}{\mu}v_y\right)u_{xx} + \left(2v_x + \frac{(\mu^2 + 1)}{\mu}u_y\right)v_{xx} \\ &+ \left(2u_y + \frac{(\mu^2 + 1)}{\mu}v_x\right)u_{xy} + \left(2v_y - \frac{(\mu^2 + 1)}{\mu}u_x\right)v_{xy} \\ &+ \alpha_x - \beta_x, \end{aligned} \quad (5.27)$$

$$\begin{aligned} \frac{\partial H}{\partial y} &= \left(2u_y + \frac{(\mu^2 + 1)}{\mu}v_x\right)u_{yy} + \left(2v_y - \frac{(\mu^2 + 1)}{\mu}u_x\right)v_{yy} \\ &+ \left(2u_x - \frac{(\mu^2 + 1)}{\mu}v_y\right)u_{xy} + \left(2v_x + \frac{(\mu^2 + 1)}{\mu}u_y\right)v_{xy} \\ &+ \alpha_y - \beta_y. \end{aligned} \quad (5.28)$$

The desired  $\alpha$  and  $\beta$  functions should satisfy  $\alpha(\mu; x, y) < \beta(\mu; x, y)$ , while their gradient should guarantee that the partial derivatives of  $H$  in (5.27) and (5.28) are never simultaneously vanishing.



## VI. ENVELOPES OF OUR PLANE CURVE

Our real manifold is a plane curve  $\Gamma$  whose equation

$$H(\mu; x, y) = 0 \quad (6.1)$$

involves an arbitrary parameter  $\mu \in [0, 1[$ . As is well known, if each of the positions of the curve  $\Gamma$  is tangent to a fixed curve  $E$ , the curve  $E$  is called the envelope of the curves  $\Gamma$ , which are said to be enveloped by  $E$ .

If an envelope  $E$  exists, let  $(x, y)$  be the point of tangency of  $E$  with that one of the curves  $\Gamma$  that corresponds to a certain value  $\mu$  of the parameter. By definition, the tangents to the curves  $E$  and  $\Gamma$  coincide for all values of  $\mu$ . If  $\delta x$  and  $\delta y$  are two quantities proportional to the direction cosines of the tangent to  $\Gamma$ , and if  $\frac{dx}{d\mu}$  and  $\frac{dy}{d\mu}$  are the derivatives of the unknown functions  $x = \phi(\mu)$  and  $y = \psi(\mu)$ , the necessary condition for tangency is [25]

$$\frac{\frac{dx}{d\mu}}{\delta x} = \frac{\frac{dy}{d\mu}}{\delta y} \implies \frac{dy}{d\mu} \delta x - \frac{dx}{d\mu} \delta y = 0. \quad (6.2)$$

On the other hand, since  $\mu$  in Eq. (6.1) has a constant value for the particular curve  $\Gamma$  considered, we have

$$\frac{\partial H}{\partial x} \delta x + \frac{\partial H}{\partial y} \delta y = 0, \quad (6.3)$$

which determines the tangent to  $\Gamma$ . Moreover, the two unknown functions  $x = \phi(\mu)$ ,  $y = \psi(\mu)$  satisfy Eq. (6.1), where  $\mu$  is now the independent variable. Hence

$$\frac{dH}{d\mu} = \frac{\partial H}{\partial x} \frac{dx}{d\mu} + \frac{\partial H}{\partial y} \frac{dy}{d\mu} + \frac{\partial H}{\partial \mu} = 0. \quad (6.4)$$

The linear homogeneous system given by Eqs. (6.2) and (6.3) has nonvanishing solutions for  $\delta x$  and  $\delta y$  if and only if the determinant of the  $2 \times 2$  matrix of coefficients vanishes, i.e.

$$\frac{\partial H}{\partial x} \frac{dx}{d\mu} + \frac{\partial H}{\partial y} \frac{dy}{d\mu} = 0. \quad (6.5)$$

By virtue of Eqs. (6.4) and (6.5) one finds that, if an envelope exists, its equation can be found by eliminating the parameter  $\mu$  between the equations  $H = 0$  and

$$\frac{\partial H}{\partial \mu} = 0. \quad (6.6)$$

If  $R(x, y) = 0$  is the equation obtained by eliminating  $\mu$  between (6.1) and (6.6), it can be shown that it represents either the envelope of the curves  $\Gamma$  or the locus of singular points

of these curves, at which

$$\left\{ H(\mu; x, y) = 0, \frac{\partial H}{\partial x} = 0, \frac{\partial H}{\partial y} = 0 \right\} \implies \frac{\partial H}{\partial \mu} = 0. \quad (6.7)$$

In other words, the plane curve  $R(x, y) = 0$  consists of two disjoint parts, one of which is the envelope, while the other is the locus of singular points.

When Eq. (6.1) is taken to have the form (5.26), we find that the curve of equation  $R(x, y) = 0$  of the general theory outlined before can be obtained, at least in principle, from

$$H = 0 \implies (u_x)^2 + (v_x)^2 + (u_y)^2 + (v_y)^2 = \left( \mu + \frac{1}{\mu} \right) (u_x v_y - u_y v_x) + \beta(\mu; x, y) - \alpha(\mu; x, y), \quad (6.8)$$

$$\frac{\partial H}{\partial \mu} = 0 \implies \left( 1 - \frac{1}{\mu^2} \right) (u_x v_y - u_y v_x) + \frac{\partial \beta}{\partial \mu} - \frac{\partial \alpha}{\partial \mu} = 0. \quad (6.9)$$

Interestingly, the construction of envelopes for the plane curve associated to pseudoholomorphic functions of a complex variable leads to the elimination of the parameter  $\mu \in [0, 1[$  provided one is able to solve the nonlinear partial differential equations (6.8) and (6.9).

#### A. The locus of points where $H, \frac{\partial H}{\partial \mu}, \frac{\partial^2 H}{\partial \mu^2}$ all vanish

Let us now consider a particular plane curve

$$H(\mu_1; x, y) = 0 \quad (6.10)$$

and a nearby curve, resulting from a small change of the parameter, i.e.

$$H(\mu_1 + \lambda; x, y) = 0. \quad (6.11)$$

This equation can be replaced by

$$\frac{H(\mu_1 + \lambda; x, y) - H(\mu_1; x, y)}{(\mu_1 + \lambda) - \mu_1} = 0, \quad (6.12)$$

which converges, as  $\lambda \rightarrow 0$ , to the limit equation

$$\left( \frac{\partial H}{\partial \mu} \right)_{\mu=\mu_1} = 0. \quad (6.13)$$

Inspired by what is done in Ref. [26] in the case of surfaces, we may therefore consider the solution set of the three simultaneous equations

$$H = 0, \frac{\partial H}{\partial \mu} = 0, H(\mu + \lambda; x, y) = 0, \quad (6.14)$$

the latter of which can be written in the form

$$H + \lambda \frac{\partial H}{\partial \mu} + \frac{\lambda^2}{2} \frac{\partial^2 H}{\partial \mu^2} + \eta = 0, \quad (6.15)$$

where  $\eta = O(\lambda^3)$ . By construction, this scheme is equivalent to studying the equations

$$H = 0, \quad \frac{\partial H}{\partial \mu} = 0, \quad \frac{\partial^2 H}{\partial \mu^2} + 2 \frac{\eta}{\lambda^2} = 0, \quad (6.16)$$

where the latter equation converges, as  $\lambda \rightarrow 0$ , to the limit equation

$$\frac{\partial^2 H}{\partial \mu^2} = 0. \quad (6.17)$$

In our problem, our Eqs. (6.8) and (6.9) should be supplemented by

$$\frac{\partial^2 H}{\partial \mu^2} = 0 \implies \frac{2}{\mu^3} (u_x v_y - u_y v_x) + \frac{\partial^2 \beta}{\partial \mu^2} - \frac{\partial^2 \alpha}{\partial \mu^2} = 0. \quad (6.18)$$

This scheme has interesting potentialities because, by virtue of (6.9), we can write Eq. (6.18) in the form

$$\frac{2}{\mu(1-\mu^2)} \left( \frac{\partial \beta}{\partial \mu} - \frac{\partial \alpha}{\partial \mu} \right) + \frac{\partial^2 \beta}{\partial \mu^2} - \frac{\partial^2 \alpha}{\partial \mu^2} = 0, \quad (6.19)$$

which implies that, for a suitable unknown function  $\sigma : x, y \rightarrow \sigma(x, y)$ , one can write

$$\gamma \equiv \frac{\partial \beta}{\partial \mu} - \frac{\partial \alpha}{\partial \mu} = \left( \frac{1}{\mu^2} - 1 \right) \sigma(x, y). \quad (6.20)$$

Note that, in full agreement with what we say elsewhere in our paper, our formulas are nontrivial only if  $\mu \neq 1$ .

## VII. CONCLUDING REMARKS

In our paper, we have defined in Sect. 3 the concept of pseudoholomorphic function of  $n$  complex variables. Moreover, in the case of one complex dimension, we have shown in Sect. 5 that the differential inequalities describing pseudoholomorphicity can be used to define a one-real-dimensional manifold (by the vanishing of a function with nonzero gradient), which is here a 1-parameter family of plane curves. In particular, if the functions  $u$  and  $v$  satisfying these equations are taken to be of the form (5.8) and (5.9), our construction becomes equivalent to obtaining a positive-definite quadratic form in the  $(x, y)$  variables, for which (5.11)-(5.13), (5.20) and (5.21) should hold. On studying the envelopes associated to our plane curves, the parameter of the general theory can be eliminated by solving

two nonlinear partial differential equations. As far as we know, the consideration of such properties never appeared before in the literature, although the basic concepts of our analysis were all well-known, when taken separately.

As far as we can see, no confusion should arise with the construction of pseudo-holomorphic submanifolds performed in Ref. [27], where the author starts instead from a compact, connected 4-manifold  $X$  with a symplectic form  $\omega$ , and considers an almost complex structure  $J$  for the tangent bundle  $TX$ ,  $J$  being chosen to be compatible with the form  $\omega$ . Given a compact submanifold  $\Sigma$  of  $X$ ,  $\Sigma$  is said to be pseudo-holomorphic when  $J$  maps the tangent bundle  $T\Sigma$  to itself as a subspace of  $TX|_{\Sigma}$ .

It would be interesting to build one-complex-dimensional manifolds whose transition functions are pseudoholomorphic according to the global theory of our Secs. 2 and 3, but the differential inequalities involved made it difficult for us to make progress along this line. In the present paper we have instead focused on one-real-dimensional manifolds associated to the differential inequalities (2.23) of the global theory. Our next goal will be to get rid of the unknown parameter appearing in the definition of the real manifold by studying the geometry of plane curves, which will be reported in a future work [28].

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### Appendix A: Counterexamples

Suppose that, for simplicity, we try to reduce the differential inequalities of Secs. 2 and 3 to equalities. For the two-variable case, for  $k = 1$  if we saturate the first of the inequalities (3.23) we obtain two independent equalities

$$\mu_1 J_1 = \varphi_1^2, \quad \Phi_1^2 = \frac{J_1}{\mu_1} \tag{A1}$$

Now from (A1) one can see that the ratio of the squares of upper and lower limits of the increment ratios

$$\frac{\Phi_1^2}{\varphi_1^2} = \frac{1}{\mu_1^2}, \quad (\text{A2})$$

which in turn yields for  $\mu_1 < 1$

$$\Phi_1^2 > \varphi_1^2, \quad (\text{A3})$$

and this minorization is consistent with our definitions.

We note that we can form a nonlinear partial differential equation involving  $u_{x_1}, u_{y_1}, v_{x_1}, v_{y_1}$  by using equations (3.21) and (A1) and writing

$$\mu_1 J_1 = \frac{1}{2}(E_1 + G_1 - \omega_1) \quad (\text{A4})$$

where

$$E_1 \equiv (u_{x_1})^2 + (v_{x_1})^2, \quad G_1 \equiv (u_{y_1})^2 + (v_{y_1})^2, \quad F_1 \equiv u_{x_1}u_{y_1} + v_{x_1}v_{y_1}, \quad (\text{A5})$$

and

$$\omega_1 \equiv \sqrt{(E_1 - G_1)^2 + 4F_1^2}. \quad (\text{A6})$$

Since  $J_1 \neq 0$ , after a little algebra one can find the following differential equation:

$$(\mu_1 + 1)(u_{x_1}v_{y_1} - u_{y_1}v_{x_1}) - [(u_{x_1})^2 + (u_{y_1})^2 + (v_{x_1})^2 + (v_{y_1})^2]\mu_1 = 0, \quad (\text{A7})$$

where we want to solve for  $u(x_1, y_1)$  and  $v(x_1, y_1)$ .

From here onwards we will be writing  $x_1 = x$  and  $y_1 = y$  and the parameter  $\alpha = \alpha_{\mu_1} = 1 + \frac{1}{\mu_1}$  so that we settle for this partial differential equation

$$\alpha(u_x v_y - u_y v_x) = (u_x)^2 + (u_y)^2 + (v_x)^2 + (v_y)^2. \quad (\text{A8})$$

Now for  $\mu_1 = 1$  i.e.  $\alpha = 2$  we get the class of holomorphic functions with Cauchy-Riemann conditions being satisfied, since in this case one can write (A7) as

$$(u_x - v_y)^2 + (u_y + v_x)^2 = 0.$$

In order to get the most general solution for the class of pseudo analytic functions we need to solve for

$$(\alpha - 2)(u_x v_y - u_y v_x) = (u_x - v_y)^2 + (u_y + v_x)^2. \quad (\text{A9})$$

As  $\mu_1 = \frac{1}{2}$  and hence  $\alpha = 3$  is an admissible choice we might try to find a solution for the following nonlinear PDE in 2 variables with constant coefficients

$$(u_x v_y - u_y v_x) = (u_x - v_y)^2 + (u_y + v_x)^2. \quad (\text{A10})$$

But even before solving this, if we try to saturate the other inequality in (3.23) and work out the case  $\mu_1 = \frac{1}{2}$  we soon run into inconsistencies as we end up getting an equation which is

$$\frac{1}{2}(u_x v_y - u_y v_x) = (u_x - v_y)^2 + (u_y + v_x)^2. \quad (\text{A11})$$

This clearly hints at possible inconsistencies that we head towards with our attempt of saturating the set of inequalities. The conclusion is that we need strict inequalities in order to deal with pseudoholomorphic functions.

As a second example, let  $w$  be the holomorphic function

$$w(z) = z^2 = u(x, y) + iv(x, y) \quad (\text{A12})$$

so that, by definition, the functions  $u(x, y), v(x, y)$  read as

$$u(x, y) = (x^2 - y^2), \quad v(x, y) = 2xy. \quad (\text{A13})$$

Now let us try to deform  $w(z)$  by adding to it a small non-holomorphic part  $\epsilon \bar{z}$  such that the modified functions  $u(x, y), v(x, y)$  become

$$u(x, y) = (x^2 - y^2 + \epsilon x), \quad v(x, y) = (2xy - \epsilon y). \quad (\text{A14})$$

With this it is easy to see that the Cauchy-Riemann condition breaks down as  $u_x \neq v_y$  although  $u_y = -v_x$ . This is clearly a non-analytic case, but can we go ahead with this and recover, with a suitable choice of  $\mu < 1$ , the pseudoholomorphic case?

Let us present a few computational details. The Jacobian defined in (2.2) for this case becomes

$$J(x, y) = 4(x^2 + y^2) - \epsilon^2. \quad (\text{A15})$$

If we make use of the master inequality (2.5) we find that, for a positive real number  $\mu \in ]0, 1]$ , it is always true that

$$\varphi^2(x, y) \geq \mu^2 \Phi^2(x, y). \quad (\text{A16})$$

Using the identities (2.21) we get a ratio satisfying the following bound  $\forall \mu$ :

$$\frac{E + G - \omega}{E + G + \omega} \geq \mu^2, \quad (\text{A17})$$

with  $E$ ,  $G$  and  $\omega$  respectively given by

$$\begin{aligned} E &= (u_x)^2 + (v_x)^2 = (2x + \epsilon)^2 + 4y^2, \\ G &= (u_y)^2 + (v_y)^2 = 4y^2 + (2x - \epsilon)^2, \\ \omega &= \sqrt{(E + G)^2 - 4J^2} = 8\sqrt{(x^2 + y^2)}\epsilon. \end{aligned} \quad (\text{A18})$$

Upon using (A16) and (A17),  $\forall \epsilon$  small we get the following bound on  $\epsilon$ :

$$2(x^2 + y^2) > \epsilon^2.$$

However, since  $\epsilon$  is very small but finite, as  $(x, y)$  tends to  $(0, 0)$  this condition is violated on a set of finite measure because the Lebesgue measure of the set

$$A_\epsilon = \{x, y : 0 \leq 2(x^2 + y^2) < \epsilon^2\}$$

is nonvanishing. But then the condition for pseudoholomorphicity would not be satisfied almost everywhere. This means that our example fails to provide a pseudoholomorphic function.

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